CLOSED-CONSTRUCTIBLE FUNCTIONS ARE PIECE-WISE CLOSED

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Abstract. A subset $B \subset Y$ is constructible if it is an element of the smallest family that contains all open sets and is stable under finite intersections and complements. A function $f : X \to Y$ is said to be piece-wise closed if X can be written as a countable union of closed sets Z_n such that f is closed on every Z_n. We prove that if a continuous function f takes each closed set into a constructible subset of Y, then f is piece-wise closed.

All spaces in this paper are supposed to be separable and metrizable, and all the functions are supposed to be continuous and onto.

A subset $B \subset Y$ is *constructible* if it is an element of the smallest family that contains all open sets and is stable under finite intersections and complements.

In topology, a constructible set is a finite union of locally closed sets (a set is *locally closed* or is an LC_1-set if it is the intersection of an open set and a closed set).

In algebraic geometry, a constructible set is any zero set of a system of polynomial equations and inequations. A function f is said to be *closed–constructible* (resp., *closed–F_σ*) if f takes closed sets into constructible (resp., F_σ) ones. It is clear that every closed–constructible function is closed–F_σ.

A function $f : X \to Y$ is said to be *piece-wise closed*if X can be written as a countable union of closed sets Z_n such that f is closed on every Z_n.

Hansell, Rogers and Jayne gave a corrected form of their previous result [6, Theorem 1], [7] using additional hypotheses $(a) - (d)$ [4, Theorem 3].

Under hypothesis (a) (=Fleissner's axiom, which is consistent with the usual axiom ZFC), they established the correctness of their first conclusion [6][Lemma 2]: each continuous, closed-F_σ function between absolute Souslin sets X and Y is piece-wise closed.

Unfortunately, the theory above is not sufficient for important applications such as the case of closed-LC_1 functions between non-Souslin subsets of the real line $\mathbf{R}$.

Motivated by this observation, we study the extensions of the theory of closed–Borel functions, whose major case is closed-LC_2 functions (a simple case for such non-continuous functions was recently considered in [10]).

We obtain the following main theorem:

Theorem 1. *Every closed–constructible function is piece-wise closed.*

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Theorem 1 immediataely implies the following:

Corollary 1. *Each constructible-measurable, closed (or open), one-to-one function is piece-wise continuous.*

Note that this result is different from that of Banakh and Bokalo [2], which states that a constructible-measurable function of hereditary Baire space X is piece-wise continuous.

Finally, we give some simple observations to the hypothesis (d) mentioned in the beginning: each preimage $f^{-1}(y)$ of points from Y is compact.

This hypothesis looks fairly strong: as we demonstrated in simple Proposition 3, a function f under hypothesis (d) becomes a closed–F_σ function. However, in the same situation (Example 3), such a conclusion becomes false if the requirement (d) is weakened to the following: each preimage $f^{-1}(y)$ of points from Y is completely metrizable.

Proposition 1. *Let $f : X \to Y$ be a continuous function with compact fibers and f take elements of a clopen base $\mathcal{B}$ into closed sets. Then f is a closed and, hence, closed–constructible function.*

Note that in following Example 1 all preimages $f^{-1}(y)$ of points from Y are completely metrizable, but not compete under the given metric.

Example 1. *There exists a continuous and open function $f : I_X \to I_Y$ from Polish spaces $I_X, I_Y \subset \boldsymbol{C}$ that takes elements of some clopen base of I_X into clopen sets, but f is not closed–F_σ.*

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